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**Pre-Leaving Certificate Examination, 2014**

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# **Applied Mathematics**

## **Marking Scheme**

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**Ordinary**      Pg. 2

**Higher**        Pg. 19

ExamCentre,  
Units 3/4,  
Fonthill Business Park,  
Fonthill Road,  
Dublin 22.

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Tel: (01) 616 62 62  
Fax: (01) 616 62 63  
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## Applied Mathematics

Ordinary Level  
Marking Scheme (300 marks)

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### General Instructions

1. Penalties of three types are applied to students' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:  
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (Att. 2), 10 (Att. 3)

2. Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.
3. Mark scripts in red unless a student uses red. If a student uses red, mark the script in blue or black.
4. Number the grid on each script 1 to 9 in numerical order, not the order of answering.
5. Scrutinise **all** pages of the answer book.
6. The marking scheme shows one correct solution to each question. In many cases, there are other equally valid methods.
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**Six questions to be answered. All questions carry equal marks. (6 × 50m)**

1. Four points,  $A$ ,  $B$ ,  $C$  and  $D$  lie on a straight level road. A car, travelling with uniform acceleration, passes  $A$ , and then 2 seconds later it passes  $B$  with a speed of  $16 \text{ m s}^{-1}$ . Three seconds after the car passes  $B$ , it passes  $C$  which is 66 m from  $B$ . On passing  $C$ , the car immediately decelerates uniformly to rest at  $D$ . The total distance from  $A$  to  $D$  is 160 m.

Find (i) the uniform acceleration of the car (10)

For  $[BC]$ ,  $u = 16$ ,  $s = 66$ ,  $t = 3$

$$s = ut + \frac{1}{2}at^2$$

$$66 = (16)(3) + \frac{1}{2}a(3)^2 \quad \dots (5\text{m})$$

$$66 = 48 + \frac{9}{2}a$$

$$\frac{9}{2}a = 18$$

$$9a = 36$$

$$a = 4 \text{ ms}^{-2} \quad \dots (5\text{m})$$

(ii) the speed of the car at  $A$  (10)

For  $[AB]$ ,  $v = 16$ ,  $a = 4$ ,  $t = 2$

$$v = u + at$$

$$16 = u + (4)(2) \quad \dots (5\text{m})$$

$$16 = u + 8$$

$$u = 8 \text{ ms}^{-1} \quad \dots (5\text{m})$$

(iii) the speed of the car at  $C$  (5)

For  $[BC]$ ,  $u = 16$ ,  $a = 4$ ,  $t = 3$

$$v = u + at$$

$$v = (16) + (4)(3)$$

$$v = 28 \text{ ms}^{-1} \quad \dots (5\text{m})$$

(iv)  $|AB|$ , the distance from  $A$  to  $B$  (10)

For  $[AB]$ ,  $u = 8$ ,  $v = 16$ ,  $a = 4$

$$v^2 = u^2 + 2as$$

$$(16)^2 = (8)^2 + 2(4)s \quad \dots (5\text{m})$$

$$256 = 64 + 8s$$

$$8s = 192$$

$$s = 24 \text{ m} \quad \dots (5\text{m})$$

(v) the time from  $C$  to  $D$ , i.e. the time of the deceleration. (15)

Distance from  $C$  to  $D$

$$= 160 - 24 - 66$$

$$= 70 \quad \dots (5\text{m})$$

For  $[CD]$ ,  $u = 28$ ,  $v = 0$ ,  $s = 70$

$$v^2 = u^2 + 2as$$

$$(0)^2 = (28)^2 + 2a(70)$$

$$0 = 784 + 140a$$

$$140a = -784$$

$$a = -\frac{28}{5} = -5.6 \quad \dots (5\text{m})$$

Then

$$v = u + at$$

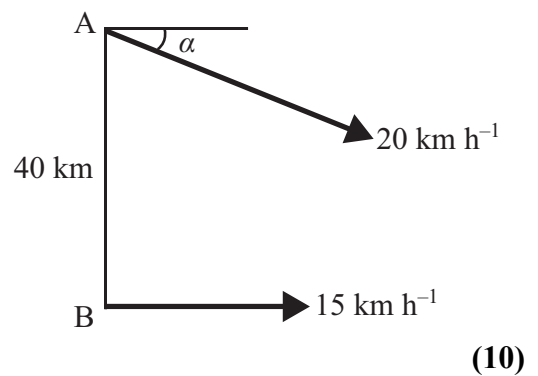
$$0 = (28) + (-5.6)t$$

$$5.6t = 28$$

$$t = 5 \text{ s} \quad \dots (5\text{m})$$

2. At noon, ship A is 40 km due north of ship B. A is moving at a constant speed of  $20 \text{ km h}^{-1}$  in the direction east  $\alpha$  south, where  $\tan \alpha = \frac{5}{12}$ .

B is moving due east at a constant speed of  $15 \text{ km h}^{-1}$ .



Find (i) the velocity of A in terms of  $\vec{i}$  and  $\vec{j}$

(10)

$$\vec{v}_A = 20 \cos \alpha \vec{i} - 20 \sin \alpha \vec{j} \quad \dots (5\text{m})$$

$$\vec{v}_A = 18.46 \vec{i} - 7.70 \vec{j} \quad \dots (5\text{m})$$

(ii) the velocity of B in terms of  $\vec{i}$  and  $\vec{j}$

(5)

$$\vec{v}_B = 15 \vec{i} \quad \dots (5\text{m})$$

(iii) the velocity of A relative to B in terms of  $\vec{i}$  and  $\vec{j}$

(10)

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B \quad \dots (5\text{m})$$

$$\vec{v}_{AB} = (18.46 \vec{i} - 7.70 \vec{j}) - (15 \vec{i})$$

$$\vec{v}_{AB} = 3.46 \vec{i} - 7.70 \vec{j} \quad \dots (5\text{m})$$

(iv) the shortest distance between A and B in the subsequent motion.

(15)

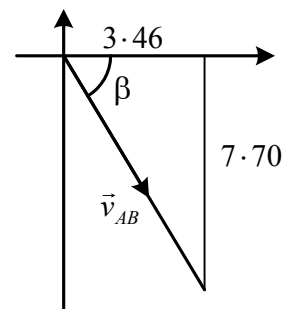
$$|\vec{v}_{AB}| = \sqrt{3.46^2 + 7.70^2}$$

$$|\vec{v}_{AB}| = 8.44 \text{ km h}^{-1}$$

and

$$\beta = \tan^{-1} \frac{7.70}{3.46} = 65.80^\circ$$

... (5m)



**Further answers overleaf**

Let  $M$  be the point of closest approach on the path of  $A$  relative to  $B$ .

Shortest distance:

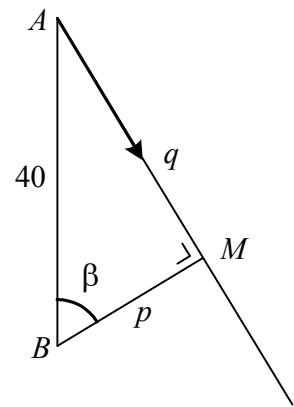
$$= p$$

$$= |BM|$$

$$= 40 \cos \beta \quad \dots (5\text{m})$$

$$= 40 \cos 65 \cdot 80^\circ$$

$$= 16 \cdot 40 \text{ km} \quad \dots (5\text{m})$$



- (v) the time, to the nearest minute, at which the ships are nearest to each other. (10)

Distance travelled by  $A$  relative to  $B$

$$= q$$

$$= |AM|$$

$$= \sqrt{40^2 - 16 \cdot 40^2}$$

$$= 36 \cdot 48 \text{ km} \quad \dots (5\text{m})$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{36 \cdot 48}{8 \cdot 44}$$

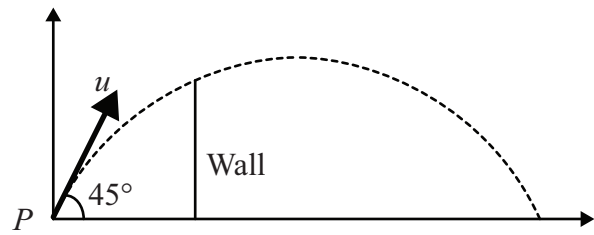
$$= 4 \cdot 32 \text{ hours}$$

$$= 4 \text{ hours } 19 \text{ minutes} \quad \dots (5\text{m})$$

3. A particle is projected from a point  $P$  on horizontal ground with an initial speed of  $u \text{ m s}^{-1}$  at an angle of  $45^\circ$  to the horizontal.

The particle strikes the ground  $5\sqrt{2}$  seconds later.

Two seconds after being projected, the particle just clears the top of a vertical wall.



Find (i) the value of  $u$  (15)

$$\text{For the time of flight, } t = 5\sqrt{2} \text{ and } s_y = 0 \quad \dots (5\text{m})$$

$$u \sin 45^\circ (5\sqrt{2}) - \frac{1}{2}(10)(5\sqrt{2})^2 = 0 \quad \dots (5\text{m})$$

$$u \left( \frac{1}{\sqrt{2}} \right) (5\sqrt{2}) - 250 = 0$$

$$5u = 250$$

$$u = 50 \quad \dots (5\text{m})$$

(ii) the range of the particle on the horizontal ground (10)

$$\text{Range} = (50) \cos 45^\circ (5\sqrt{2}) \quad \dots (5\text{m})$$

$$= 50 \left( \frac{1}{\sqrt{2}} \right) (5\sqrt{2})$$

$$= 250 \text{ m} \quad \dots (5\text{m})$$

(iii) the greatest height reached by the particle (15)

$$\text{Time to greatest height} = \frac{1}{2}(5\sqrt{2}) = \frac{5}{\sqrt{2}} \quad \dots (5\text{m})$$

$$\text{Greatest height} = (50) \sin 45^\circ \left( \frac{5}{\sqrt{2}} \right) - \frac{1}{2}(10) \left( \frac{5}{\sqrt{2}} \right)^2 \quad \dots (5\text{m})$$

$$= \frac{250}{2} - \frac{125}{2}$$

$$= 62.5 \text{ m} \quad \dots (5\text{m})$$

**Further answers overleaf**

- (iv) the height of the vertical wall, correct to two decimal places. (10)

When  $t = 2$ ,

$$\text{height of wall} = (50)\sin 45^\circ(2) - \frac{1}{2}(10)(2)^2 \quad \dots (5\text{m})$$

$$= 50\sqrt{2} - 20$$

$$= 50 \cdot 71 \text{ m} \quad \dots (5\text{m})$$



4. (a) Two particles of masses 3 kg and 5kg are connected by a taut, light, inextensible string which passes over a smooth, light pulley.

The system is released from rest.

Find (i) the common acceleration of the particles

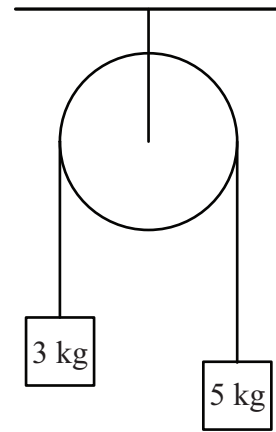
$$T - 3g = 3a \quad \dots (5m)$$

$$5g - T = 5a \quad \dots (5m)$$

$$2g = 8a$$

$$a = \frac{g}{4}$$

$$a = 2.5 \text{ ms}^{-2} \quad \dots (5m)$$



(15)

(ii) the tension in the string.

(5)

$$T = 3a + 3g$$

$$T = 3(2.5) + 3(10)$$

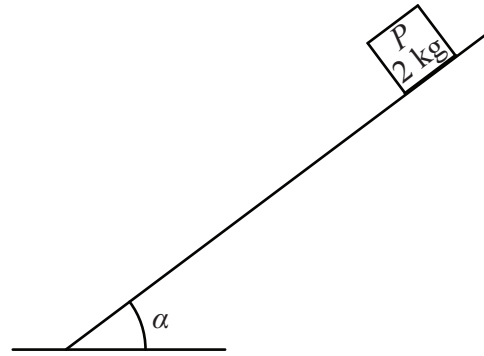
$$T = 37.5 \text{ N} \quad \dots (5m)$$

**Further answers overleaf**

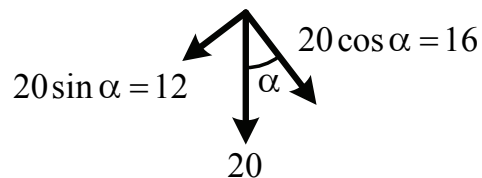
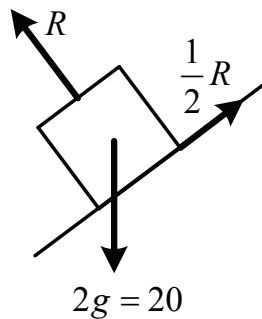
- (b) A particle,  $P$ , of mass 2 kg sits on a rough plane inclined at an angle  $\alpha$  to the horizontal, where  $\tan \alpha = \frac{3}{4}$ .

The coefficient of friction between  $P$  and the plane is  $\frac{1}{2}$ .

$P$  is released from rest, and travels 4 m to the bottom of the plane.



- Find (i) the acceleration of  $P$  down the plane. (25)



... (10 m)

$$R = 16 \quad \dots (5m)$$

$$12 - \frac{1}{2}R = 2a \quad \dots (5m)$$

$$12 - 8 = 2a$$

$$4 = 2a$$

$$a = 2 \text{ ms}^{-2} \quad \dots (5m)$$

- (ii) the time taken by  $P$  to reach the bottom of the plane. (5)

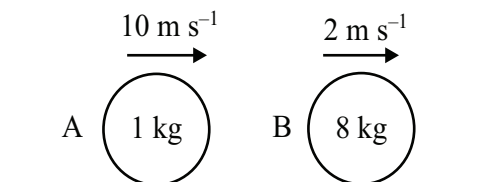
$$s = ut + \frac{1}{2}at^2$$

$$4 = (0)t + \frac{1}{2}(2)t^2$$

$$4 = t^2$$

$$t = 2 \text{ s} \quad \dots (5m)$$

5. A smooth sphere A, of mass 1 kg, collides directly with another smooth sphere B, of mass 8 kg, on a smooth horizontal table.



A and B are moving in the same direction with speeds of  $10 \text{ m s}^{-1}$  and  $2 \text{ m s}^{-1}$ , respectively.

The coefficient of restitution for the collision is  $\frac{1}{8}$ .

Find (i) the speed of A and the speed of B after the collision (30)

PCM:  $(1)(10) + (8)(2) = (1)v_1 + (8)v_2$  ... (10m)

$$26 = v_1 + 8v_2$$

$$v_1 + 8v_2 = 26$$

NEL:  $v_1 - v_2 = -\frac{1}{8}(10 - 2)$  ... (10m)

$$v_1 - v_2 = -1$$

$$-v_1 + v_2 = 1$$

Thus  $9v_2 = 27$

$$v_2 = 3 \text{ m s}^{-1}$$

and  $v_1 = 2 \text{ m s}^{-1}$  ... (10m)

(ii) the loss in kinetic energy of A due to the collision (15)

KE of A before collision  $= \frac{1}{2}(1)(10)^2 = 50$  ... (5m)

KE of A after collision  $= \frac{1}{2}(1)(2)^2 = 2$  ... (5m)

Loss in KE of A  $= 50 - 2 = 48 \text{ J}$  ... (5m)

(iii) the magnitude of the impulse imparted to B due to the collision. (5)

Impulse  $= |(8)(3) - (8)(2)| = 8 \text{ N s}$  ... (5m)

6. (a) Particles of weight 5 N, 4 N, 2 N and 3 N are placed at the points  $(1, q)$ ,  $(q, p)$ ,  $(p, 6)$  and  $(q + 1, q)$ , respectively.

The co-ordinates of the centre of gravity of the system are  $(p, q)$ .

- Find (i) the value of  $q$   
(ii) the value of  $p$ . (25)

$$p = \frac{5(1) + 4(q) + 2(p) + 3(q+1)}{5 + 4 + 2 + 3} \quad \dots (10\text{m})$$

$$14p = 5 + 4q + 2p + 3q + 3$$

$$12p - 7q = 8$$

and

$$q = \frac{5(q) + 4(p) + 2(6) + 3(q)}{5 + 4 + 2 + 3} \quad \dots (10\text{m})$$

$$14q = 5q + 4p + 12 + 3q$$

$$-4p + 6q = 12$$

then

$$12p - 7q = 8$$

$$\underline{-12p + 18q = 36}$$

$$11q = 44$$

$$q = 4$$

and

$$12p - 28 = 8$$

$$12p = 36$$

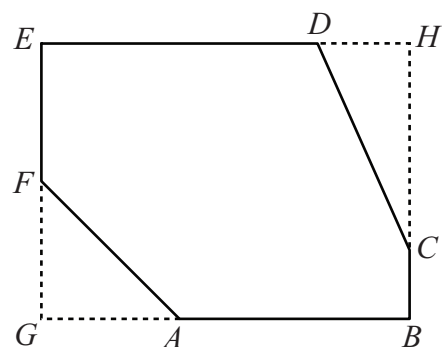
$$p = 3$$

... (5m)

- (b) A uniform lamina  $ABCDEF$  consists of a rectangle  $GBHE$  from which the two triangles  $GAF$  and  $CHD$  have been removed.

The co-ordinates of the points are  $A(6, 0)$ ,  $B(15, 0)$ ,  $C(15, 3)$ ,  $D(12, 12)$ ,  $E(0, 12)$ ,  $F(0, 6)$ ,  $G(0, 0)$  and  $H(15, 12)$ .

Find the co-ordinates of the centre of gravity of the lamina  $ABCDEF$ .



(25)

	Area	Centre of gravity	
<i>GBHE</i>	$15 \times 12 = 180$	$(7.5, 6)$	... (5m)
<i>GAF</i>	$\frac{1}{2}(6)(6) = 18$	$(2, 2)$	... (5m)
<i>CHD</i>	$\frac{1}{2}(3)(9) = 13.5$	$(14, 9)$	... (5m)
<i>ABCDEF</i>	$180 - 18 - 13.5$ $= 148.5$	$(x, y)$	

then

$$7.5 = \frac{18(2) + 13.5(14) + 148.5(x)}{180}$$

$$1350 = 36 + 189 + 148.5x$$

$$148.5x = 1125$$

$$x = 7.58$$

... (5m)

and

$$6 = \frac{18(2) + 13.5(9) + 148.5(y)}{180}$$

$$1080 = 36 + 121.5 + 148.5y$$

$$148.5y = 922.5$$

$$y = 6.21$$

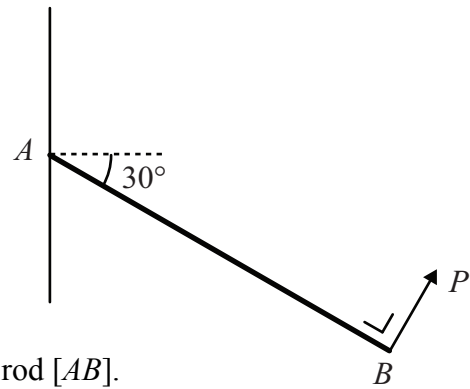
... (5m)

Thus the co-ordinates of the centre of gravity are  $(7.58, 6.21)$ .

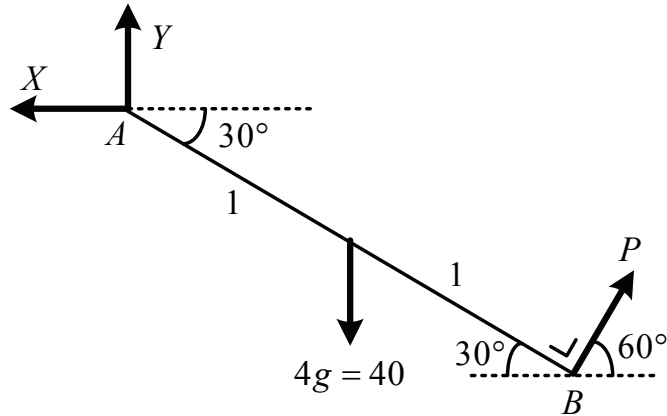
7. A uniform rod,  $[AB]$ , of weight 40 N and length 2 m, is smoothly hinged at end  $A$  to a vertical wall.

The rod is held inclined at  $30^\circ$  below the horizontal by a force  $P$  applied at  $B$ .

The force  $P$  is at right angles to the rod  $[AB]$ .



- (i) Show on a diagram all the forces acting on the rod  $[AB]$ . (15)



... (15m)

- (ii) Find the magnitude of  $P$ . (15)

Moments about A:

Anticlockwise Moments = Clockwise Moments

$$P(2) = 40(1 \cos 30^\circ) \quad \dots (5m, 5m)$$

$$P = 20 \left( \frac{\sqrt{3}}{2} \right)$$

$$P = 10\sqrt{3} \text{ N} \quad \dots (5m)$$

(iii) Find the magnitude of the reaction at the hinge,  $A$ . (20)

The rod is in equilibrium.

$$\uparrow = \downarrow : \quad Y + \frac{\sqrt{3}P}{2} = 40 \quad \dots(5m)$$

$$\leftarrow = \rightarrow : \quad X = \frac{P}{2} \quad \dots (5m)$$

thus

$$Y + \frac{\sqrt{3}}{2}(10\sqrt{3}) = 40$$

$$Y + 15 = 40$$

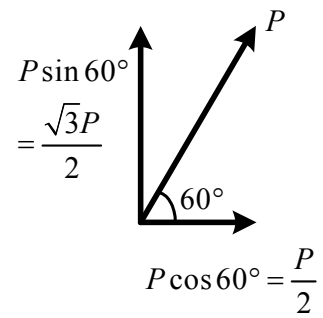
$$Y = 25$$

and

$$X = \frac{1}{2}(10\sqrt{3}) = 5\sqrt{3} \quad \dots (5m)$$

The magnitude of the reaction force at hinge  $A$  is

$$\begin{aligned} \sqrt{X^2 + Y^2} &= \sqrt{(5\sqrt{3})^2 + 25^2} \\ &= \sqrt{75 + 625} \\ &= \sqrt{700} \\ &= 10\sqrt{7} = 26.46 \text{ N} \end{aligned} \quad \dots (5m)$$



8. (a) A particle describes a horizontal circle of radius  $r$  metres with uniform angular velocity  $\omega$  radians per second. Its speed is  $12 \text{ m s}^{-1}$  and its acceleration is  $24 \text{ m s}^{-2}$ .

Find (i) the value of  $\omega$  (15)

$$\begin{aligned} \text{Speed} &= 12 \\ r\omega &= 12 && \dots(5\text{m}) \end{aligned}$$

$$\begin{aligned} \text{and} \\ \text{Acceleration} &= 24 \\ r\omega^2 &= 24 && \dots(5\text{m}) \end{aligned}$$

$$\begin{aligned} \text{then} \\ (r\omega)\omega &= 24 \\ 12\omega &= 24 \\ \omega &= 2 \text{ rad s}^{-1} && \dots(5\text{m}) \end{aligned}$$

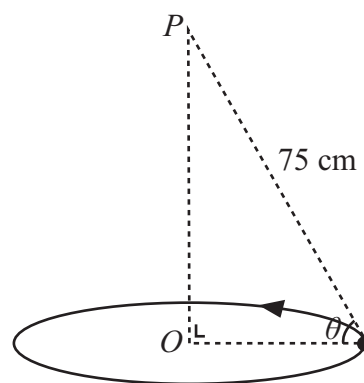
(ii) the value of  $r$ . (5)

$$\begin{aligned} \text{Then} \\ r(2) &= 12 \\ r &= 6 \text{ m} && \dots(5\text{m}) \end{aligned}$$

- (b) A particle of mass 3 kg is attached by a light, inextensible string of length 75 cm to a fixed point,  $P$ .

The particle moves in a horizontal circle, centre  $O$ , with constant speed such that the string makes an angle  $\theta$  with the horizontal,

$$\text{where } \tan \alpha = \frac{4}{3}.$$



Find (i) the value of  $r$ , the radius of the circular motion (10)

$$\begin{aligned} \tan \theta &= \frac{4}{3} \\ \cos \theta &= \frac{3}{5} \quad \text{and} \quad \sin \theta = \frac{4}{5} \end{aligned}$$

$$\frac{r}{0.75} = \cos \theta \quad \dots (5\text{m})$$

$$r = 0.75 \left( \frac{3}{5} \right) = 0.45 \text{ m} \quad \dots (5\text{m})$$

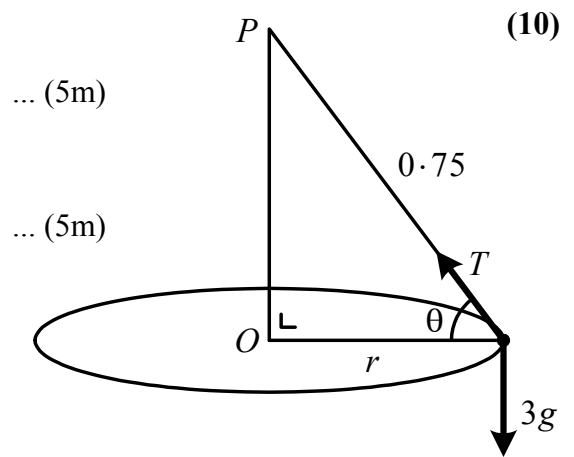


(ii) the tension in the string

$$T \sin \theta = 3g$$

$$T \left( \frac{4}{5} \right) = 30$$

$$T = 37.5 \text{ N}$$



(iii) the speed of the particle.

(10)

Circular motion:

$$T \cos \theta = \frac{mv^2}{r} \quad \dots (5m)$$

$$(37.5) \left( \frac{3}{5} \right) = \frac{(3)v^2}{0.45}$$

$$(7.5)(0.45) = v^2$$

$$v^2 = 3.375$$

$$v = 1.84 \text{ m s}^{-1} \quad \dots (5m)$$

9. (a) State the Principle of Archimedes. (10)

Statement ... (10m)

A solid piece of metal has a weight of 800 N.  
When it is completely immersed in a liquid of relative density 1.5, the metal weighs 500 N.

- Find (i) the volume of the metal (10)

$$B = \text{weight of displaced liquid}$$

$$300 = \rho Vg$$

$$300 = (1500)V(10) \quad \dots (5m)$$

$$V = 0.02 \text{ m}^3 \quad \dots (5m)$$

- (ii) the relative density of the metal. (10)

$$\text{Weight of metal} = \rho Vg$$

$$800 = \rho(0.02)(10) \quad \dots (5m)$$

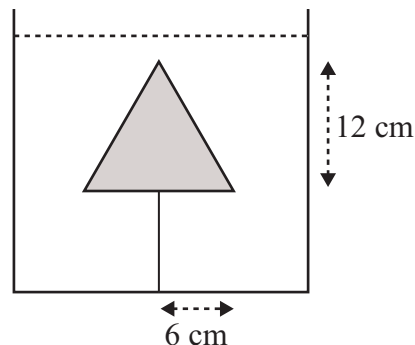
$$\rho = 4000$$

$$s = 4 \quad \dots (5m)$$

- (b) A solid cone has a base of radius 6 cm and a height of 12 cm.

The cone is made from plastic of relative density 0.81 and is completely immersed in a tank of water.

The cone is held at rest with its axis vertical by a vertical string which is attached to the centre of its base and to the base of the water tank.



- Find the tension in the string. [Density of water = 1000 kg m<sup>-3</sup>.] (20)

$$B = 1000 \left\{ \frac{1}{3} \pi (0.06)^2 (0.12) \right\} (10)$$

$$B = 4.52 \quad \dots (5m)$$

$$W = 810 \left\{ \frac{1}{3} \pi (0.06)^2 (0.12) \right\} (10)$$

$$W = 3.66 \quad \dots (5m)$$

and

$$T + W = B \quad \dots (5m)$$

$$T + 3.66 = 4.52$$

$$T = 0.86 \text{ N} \quad \dots (5m)$$

Pre-Leaving Certificate Examination, 2014

## Applied Mathematics

Higher Level  
Marking Scheme (300 marks)

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- 

**Six questions to be answered. All questions carry equal marks. (6 × 50m)**

1. (a) An underground train has to travel a distance  $d$  from rest at one station to rest at the next station. The train has a maximum acceleration of  $a$  and a maximum deceleration of  $b$ .

If the train makes the journey in the shortest possible time without having to travel at constant speed, show that the speed limit on the track,  $v$ , satisfies

$$v \geq \sqrt{\frac{2abd}{a+b}}. \quad (25)$$

If the train makes the journey without having to travel at constant speed, then it decelerates immediately after accelerating. Let  $s_1$  and  $s_2$  be the distances travelled while accelerating and decelerating, respectively. Let  $v_1$  be the maximum speed attained.

Then

$$s_1 + s_2 = d \quad \dots \mathbf{1}$$

Period of acceleration: Let  $t_1$  be the time for acceleration. Then

$$v_1 = (0) + (a)(t_1)$$

$$t_1 = \frac{v_1}{a} \quad \dots (5m)$$

and

$$s_1 = (0)(t_1) + \frac{1}{2}(a)(t_1)^2$$

$$s_1 = \frac{1}{2}a \left( \frac{v_1^2}{a^2} \right) = \frac{v_1^2}{2a} \quad \dots (5m)$$

Likewise,

$$s_2 = \frac{v_1^2}{2b}$$

From **1**,

$$\frac{v_1^2}{2a} + \frac{v_1^2}{2b} = d \quad \dots (5m)$$

$$v_1^2 \left( \frac{b+a}{2ab} \right) = d$$

$$v_1^2 = \frac{2abd}{a+b}$$

$$v_1 = \sqrt{\frac{2abd}{a+b}} \quad \dots (5m)$$

To make the journey without a period of constant speed, we need

$$v \geq v_1$$

$$v \geq \sqrt{\frac{2abd}{a+b}} \quad \dots (5m)$$

- (b) A stone is dropped from the top of a tower which is located on horizontal ground through the base of the tower. One second later, another stone is thrown vertically downwards from the same point with a speed of  $15 \text{ m s}^{-1}$ .

- (i) If the two stones reach the ground simultaneously, find the height of the tower. (15)

Let  $h$  m be the height of the tower. If the first stone takes  $t$  s to reach the ground, then the second stone will take  $(t-1)$  s.

$$\begin{aligned} \text{1st stone: } h &= (0)(t) + \frac{1}{2}(g)(t)^2 \\ h &= \frac{1}{2}gt^2 \end{aligned} \quad \dots (5\text{m})$$

$$\text{2nd stone: } h = (15)(t-1) + \frac{1}{2}g(t-1)^2 \quad \dots (5\text{m})$$

$$h = 15t - 15 + \frac{1}{2}g(t^2 - 2t + 1)$$

$$h = 15t - 15 + \frac{1}{2}gt^2 - gt + \frac{1}{2}g$$

$$\text{then } 15t - 15 + \frac{1}{2}gt^2 - gt + \frac{1}{2}g = \frac{1}{2}gt^2$$

$$15t - 15 - gt + \frac{1}{2}g = 0$$

$$(15 - g)t = 15 - \frac{1}{2}g$$

$$5 \cdot 2t = 10 \cdot 1$$

$$t = 1.94 \text{ s}$$

The height of the tower is

$$h = (4.9)(1.94)^2 = 18.44 \text{ m} \quad \dots (5\text{m})$$

- (ii) If a third stone is thrown downwards from the same point half a second after the second stone, what initial speed must it have to reach the ground at the same time as the first two stones? (10)

Let  $v \text{ m s}^{-1}$  be the initial speed of the third stone. It travels for  $0.94 - 0.5 = 0.44$  s while covering a distance of  $18.44$  m.

Thus

$$18.44 = (v)(0.44) + \frac{1}{2}(9.8)(0.44)^2 \quad \dots (5\text{m})$$

$$18.44 = 0.44v + 0.94864$$

$$0.44v = 17.49136$$

$$v = 39.75$$

The initial speed is  $39.75 \text{ m s}^{-1}$ . ... (5m)

2. (a) Two straight roads cross at right angles at  $O$ . On one road, a car is travelling due north at  $4 \text{ m s}^{-1}$ . As the car passes through  $O$ , a bus is  $50 \text{ m}$  from  $O$  and travelling east towards  $O$  at a speed of  $3 \text{ m s}^{-1}$ .

- (i) Find the velocity of the bus relative to the car.

$$\vec{v}_C = 4\vec{j}$$

$$\vec{v}_B = 3\vec{i}$$

then

$$\begin{aligned}\vec{v}_{BC} &= \vec{v}_B - \vec{v}_C \\ &= 3\vec{i} - 4\vec{j}\end{aligned}$$

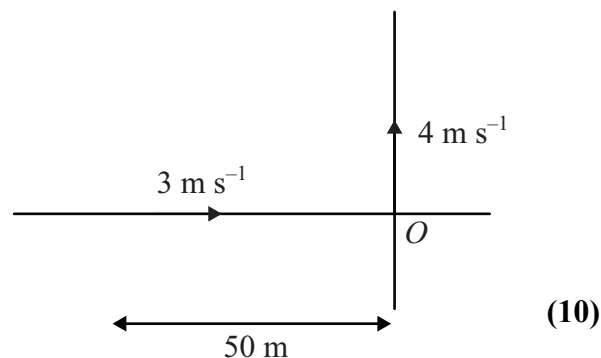
thus

$$|\vec{v}_{BC}| = \sqrt{3^2 + 4^2} = 5 \text{ m s}^{-1}$$

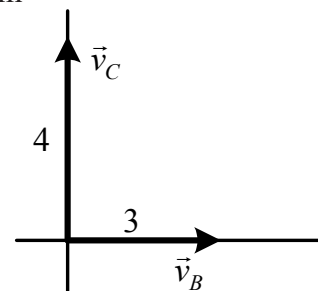
and

$$\tan \theta = \frac{4}{3}$$

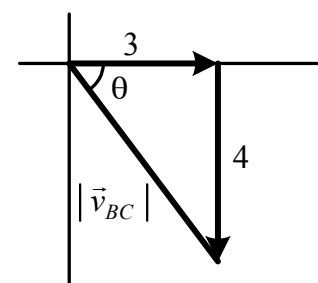
$$\theta = \tan^{-1} \frac{4}{3} = 53.13^\circ$$



... (5m)



... (5m)



- (ii) Calculate the shortest distance between the car and the bus and the time at which this occurs.

(10)

If  $p$  is the shortest distance,

$$\frac{p}{50} = \sin \theta$$

$$p = 50 \left( \frac{4}{5} \right)$$

$$p = 40 \text{ m}$$

and

$$\frac{q}{50} = \cos \theta$$

$$q = 50 \left( \frac{3}{5} \right)$$

$$q = 30 \text{ m}$$

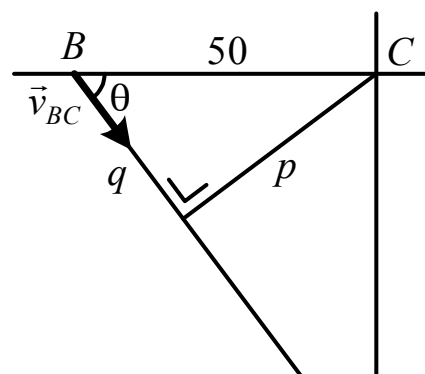
then

$$\text{time} = \frac{\text{relative distance}}{\text{relative speed}}$$

$$= \frac{30}{5}$$

$$= 6 \text{ s}$$

... (5m)



- (iii) Find the length of time during which the car and the bus are within 41 m of one another. (5)

$$x^2 = 41^2 - 40^2$$

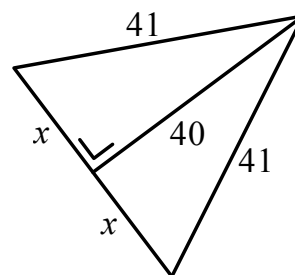
$$x^2 = 81$$

$$x = 9$$

$$\text{Distance travelled} = 2 \times 9 = 18 \text{ m}$$

$$\text{Time} = \frac{18}{5} = 3.6 \text{ s.}$$

... (5m)



- (b) A ship P is travelling in the direction  $38^\circ$  north of east. To an observer on another ship Q, which is travelling  $18^\circ$  south of east at  $33 \text{ km h}^{-1}$ , P appears to be travelling in the direction  $67^\circ$  north of west.

- (i) Find the actual speed of P, correct to one decimal place. (15)

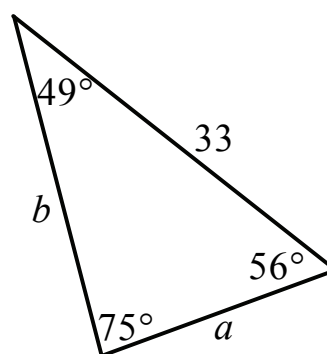
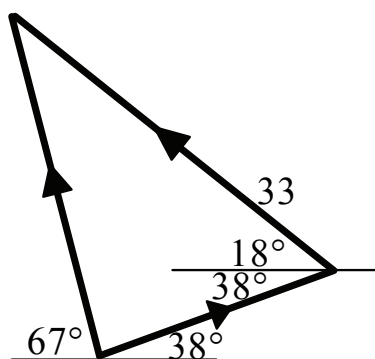


Diagram with angles:

... (10m)

Let  $a = |\vec{v}_P|$ . Then

$$\frac{a}{\sin 49^\circ} = \frac{33}{\sin 75^\circ}$$

$$a = 25.8 \text{ km h}^{-1} = |\vec{v}_P|$$

... (5m)

- (ii) Find the magnitude of the velocity of P relative to Q, correct to one decimal place. (5)

Let  $b = |\vec{v}_{PQ}|$ . Then

$$\frac{b}{\sin 56^\circ} = \frac{33}{\sin 75^\circ}$$

$$b = 28.3 \text{ km h}^{-1} = |\vec{v}_{PQ}|$$

... (5)

**Further answers overleaf**

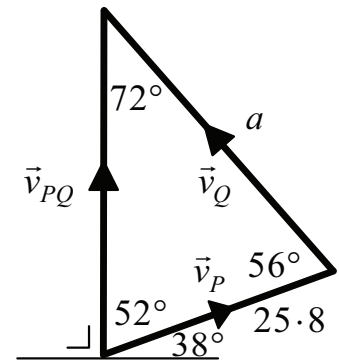
- (iii) Q reduces its speed, without changing direction, so that P appears to be travelling due north. Find the reduced speed of Q, correct to one decimal place. (5)

Let  $a$  be the new speed of Q.

$$\frac{a}{\sin 52^\circ} = \frac{25.8}{\sin 72^\circ}$$

$$a = 21.4 \text{ km h}^{-1} = |\vec{v}_Q|$$

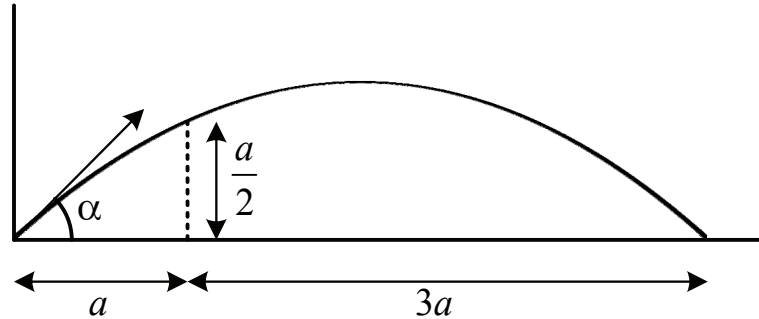
... (5)





3. (a) A particle is projected from a point on a horizontal plane so that it just clears a vertical wall of height  $\frac{a}{2}$  at a horizontal distance of  $a$  from the point of projection and strikes the plane at a horizontal distance  $3a$  **beyond the wall**.

- (i) Express the range of the particle on the horizontal plane in terms of  $a$ . (5)



From the diagram, the range is  
 $a + 3a = 4a$  ... (5m)

- (ii) If the angle of projection measured to the horizontal is  $\tan^{-1} \frac{2}{k}$ , find the value of  $k$ . (20)

Let  $\alpha$  be the angle of projection.

TOF:  $s_y = 0$

$$u \sin \alpha t - \frac{1}{2} g t^2 = 0$$

$$t = \frac{2u \sin \alpha}{g} \quad \dots (5m)$$

$$R = 4a: \quad 4a = u \cos \alpha \left( \frac{2u \sin \alpha}{g} \right)$$

$$4a = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$u^2 = \frac{2ga}{\sin \alpha \cos \alpha} \quad \dots 1 \quad \dots (5m)$$

Path contains the point  $\left( a, \frac{a}{2} \right)$ :

$$u \cos \alpha t = a, \quad u \sin \alpha t - \frac{1}{2} g t^2 = \frac{a}{2}$$

$$t = \frac{a}{u \cos \alpha}$$

**Further answers overleaf**

thus

$$u \sin \alpha \left( \frac{a}{u \cos \alpha} \right) - \frac{g}{2} \left( \frac{a^2}{u^2 \cos^2 \alpha} \right) = \frac{a}{2} \quad \dots (5m)$$

$$a \tan \alpha - \frac{ga^2}{2 \cos^2 \alpha} \left( \frac{\sin \alpha \cos \alpha}{2ga} \right) = \frac{a}{2} \quad \dots \text{by 1}$$

$$a \tan \alpha - \frac{1}{4} a \tan \alpha = \frac{a}{2}$$

$$\frac{3}{4} \tan \alpha = \frac{1}{2}$$

$$\tan \alpha = \frac{2}{3}$$

$$\alpha = \tan^{-1} \frac{2}{3}.$$

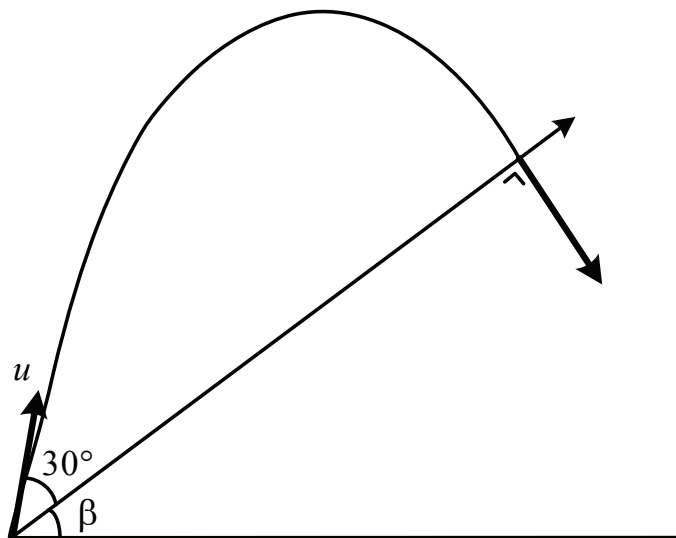
hence  $k = 3$  ... (5m)

- (b) A plane is inclined at an angle  $\beta$  to the horizontal.

A body is projected up the plane with velocity  $u \text{ m s}^{-1}$  at an angle of  $30^\circ$  to the inclined plane.

The plane of projection is vertical and contains the line of greatest slope. The particle strikes the plane at right angles.

- (i) Find the value of  $c$  and the value of  $d$  if  $\tan \beta = \frac{\sqrt{c}}{d}$ . (15)



At the same time,  $s_y = 0$  and  $v_x = 0$ .

$$s_y = 0$$

$$\frac{ut}{2} - \frac{1}{2}g \cos\beta t^2 = 0$$

$$t = \frac{u}{g \cos\beta} \quad \dots (5m)$$

At  $t = \frac{u}{g \cos\beta}$ :

$$v_x = 0$$

$$\frac{\sqrt{3}u}{2} - g \sin\beta \left( \frac{u}{g \cos\beta} \right) = 0 \quad \dots (5m)$$

$$\frac{\sqrt{3}}{2} = \tan\beta$$

thus  $c = 3$  and  $d = 2$  ... (5m)

- (ii) If the range of the body on the inclined plane is  $10\sqrt{21}$  m, find the value of  $u$ . (10)

Then

$$\text{TOF} = \frac{u}{g} \cdot \frac{\sqrt{7}}{2} = \frac{\sqrt{7}u}{2g}$$

and

$$\text{Range} = 10\sqrt{21}$$

$$\frac{\sqrt{3}u}{2} \left( \frac{\sqrt{7}u}{2g} \right) - \frac{1}{2}g \left( \frac{\sqrt{3}}{\sqrt{7}} \right) \left( \frac{7u^2}{4g^2} \right) = 10\sqrt{21} \quad \dots (5m)$$

$$\frac{\sqrt{21}u^2}{4g} - \frac{\sqrt{21}u^2}{8g} = 10\sqrt{21}$$

$$2u^2 - u^2 = 80g$$

$$u^2 = 80g$$

$$u = 28 \quad \dots (5m)$$

4. (a) A light inextensible string is attached at one end to a particle A, of mass 5 kg, hanging freely.

The string passes over a smooth fixed pulley and its other end is attached to another particle B, of mass 3 kg, which is held at a height of 5 m above the horizontal plane.

Initially the string is slack. The system is released from rest.

After B has fallen through 1 m, the string becomes taut. Find

- (i) the speed of B directly after the string becomes taut. (10)

First metre:

$$\begin{array}{ll}
 u = 0 & v^2 = u^2 + 2as \\
 v = ? & v^2 = 0^2 + 2g(1) \\
 a = g & v^2 = 2g \\
 s = 1 & v = \sqrt{2g} \qquad \dots (5m)
 \end{array}$$

Let  $v$  be the velocity of each particle after the jerk.

Momentum before = Momentum of B

$$\begin{aligned}
 &= (3)(\sqrt{2g}) \\
 &= 3\sqrt{2g}
 \end{aligned}$$

Momentum after = Momentum of A + Momentum of B

$$\begin{aligned}
 &= (5)(v) + (3)(v) \\
 &= 8v
 \end{aligned}$$

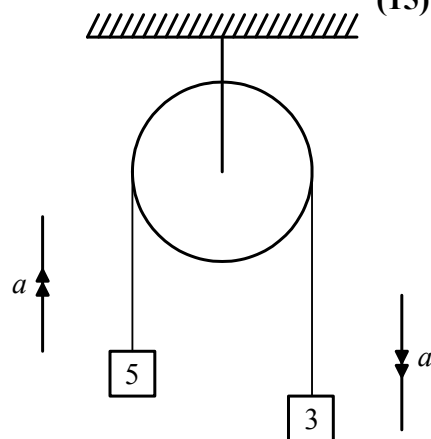
PCM: Momentum after = Momentum before

$$\begin{aligned}
 8v &= 3\sqrt{2g} \\
 v &= \frac{3\sqrt{2g}}{8} \text{ m s}^{-1} \qquad \dots (5m)
 \end{aligned}$$

- (ii) the nearest that B gets to the horizontal plane. (15)

**5 kg mass**

$$\begin{array}{ll}
 \updownarrow : & \text{Newton's 2}^{\text{nd}} \text{ Law} \\
 & 5a = T - 5g \qquad \dots 1
 \end{array}$$



**3 kg mass**

$$\updownarrow : \quad \text{Newton's 2}^{\text{nd}} \text{ Law}$$

$$3a = 3g - T \quad \dots 2$$

$$\text{Adding 1 and 2,}$$

$$8a = -2g$$

$$a = -\frac{g}{4}$$

$$u = \frac{3\sqrt{2g}}{8}$$

$$v^2 = u^2 + 2as$$

$$v = 0$$

$$0^2 = \left(\frac{3\sqrt{2g}}{8}\right)^2 + 2\left(-\frac{g}{4}\right)s$$

$$a = -\frac{g}{4}$$

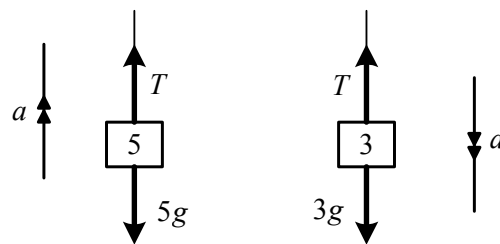
$$\frac{gs}{2} = \frac{18g}{64}$$

$$s = ?$$

$$s = \frac{36}{64}$$

$$s = \frac{9}{16} = 0.5625 \text{ m}$$

... (5m)



$$\text{Nearest B gets to the plane} = 5 - (1 + 0.5625)$$

$$= 3.4375$$

$$= 3.44 \text{ m}$$

... (5m)

(b) A wedge of mass 8 kg sits on a smooth horizontal plane.

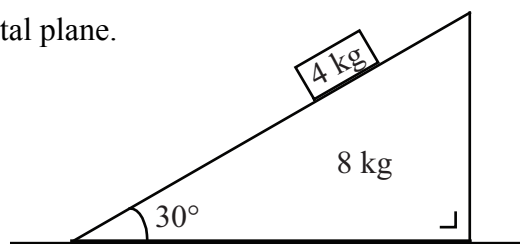
A particle of mass 4 kg sits on a face of the wedge which is inclined at  $30^\circ$  to the horizontal. The coefficient of friction

between the particle and the wedge is  $\frac{1}{2}$ .

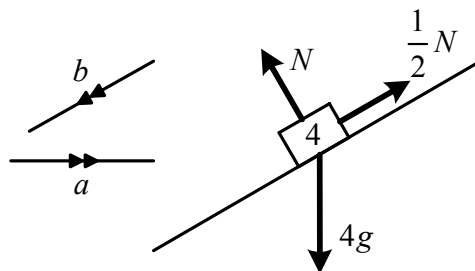
The system is released from rest.

Find the acceleration of the wedge.

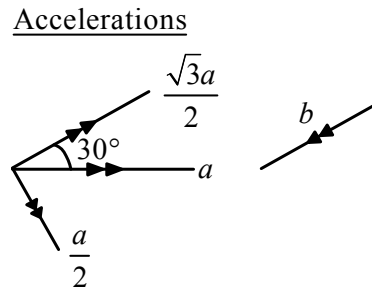
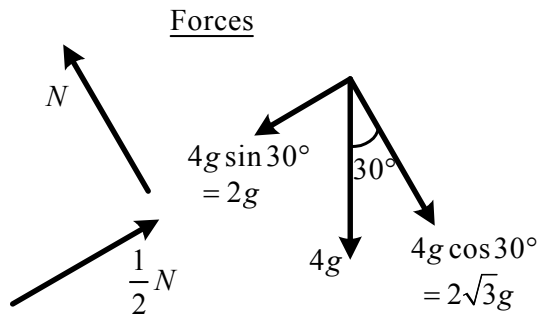
(25)



**4 kg particle**



**Further answers overleaf**



↖ : Newton's 2<sup>nd</sup> Law

$$4\left(\frac{a}{2}\right) = 2\sqrt{3}g - N \quad \dots (5m)$$

$$2a = 2\sqrt{3}g - N \quad \dots 1$$

**Wedge**

↔ : Newton's 2<sup>nd</sup> Law

$$8a = \frac{N}{2} - \frac{\sqrt{3}}{4}N \quad \dots (5m)$$

$$32a = 2N - \sqrt{3}N \quad \dots 2$$

**2 :**  $(2 - \sqrt{3})N = 32a$

$$N = \frac{32a}{2 - \sqrt{3}} \quad \dots (5m)$$

**1 :**  $2a = 2\sqrt{3}g - \frac{32a}{2 - \sqrt{3}}$  ... (5m)

$$2(2 - \sqrt{3})a = 2\sqrt{3}(2 - \sqrt{3})g - 32a$$

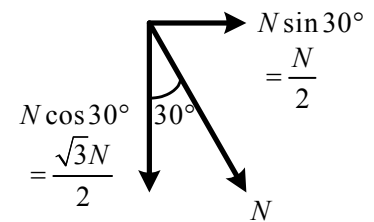
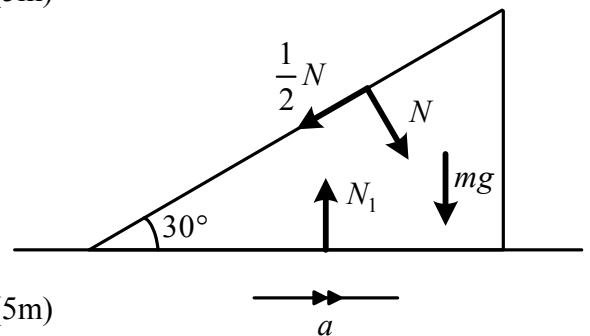
$$(4 - 2\sqrt{3} + 32)a = (4\sqrt{3} - 6)g$$

$$(36 - 2\sqrt{3})a = (4\sqrt{3} - 6)g$$

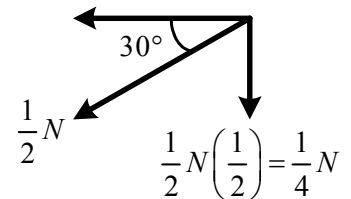
$$(18 - \sqrt{3})a = (2\sqrt{3} - 3)g$$

$$a = \frac{2\sqrt{3} - 3}{18 - \sqrt{3}} g \text{ m s}^{-2}$$

or  $a = 0.28 \text{ m s}^{-2}$  ... (5m)

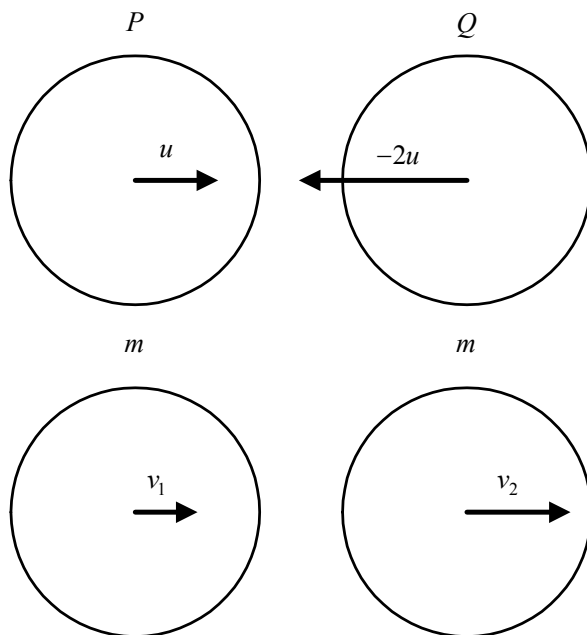


$$\frac{1}{2}N\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}N$$



5. (a) Two smooth spheres, P and Q, of equal mass, are travelling directly towards each other with speeds  $u$  and  $2u$ , respectively.

- (i) Determine if it is possible for the sphere Q to be at rest after the collision. (15)



PCM:  $mv_1 + mv_2 = mu - 2mu$

$v_1 + v_2 = -u$  ... 1

NEL:  $v_1 - v_2 = -3eu$  ... 2 ... (5m)

1 :  $v_1 + v_2 = -u$

$2 \times -1$ :  $-v_1 + v_2 = 3eu$

$2v_2 = (3e-1)u$

$v_2 = \frac{(3e-1)u}{2}$  ... (5m)

For Q to be at rest after the collision,  $v_2 = 0$ . Thus

$\frac{(3e-1)u}{2} = 0$

$3e-1 = 0$

$e = \frac{1}{3}$

As  $0 < e < 1$ , it is possible for Q to be at rest after the collision.

... (5m)

- (ii) If the fraction of the kinetic energy lost during the collision is  $\frac{27}{40}$ , find the value of  $e$ , the coefficient of restitution. (15)

Adding 1 and 2,

$$2v_1 = -(1+3e)u$$

$$v_1 = \frac{-(1+3e)u}{2}$$

Then:

$$\text{KE before impact} = \frac{1}{2}mu^2 + \frac{1}{2}m(-2u)^2 = \frac{5}{2}mu^2$$

$$\begin{aligned} \text{KE after impact} &= \frac{1}{2}mu^2 \left( \frac{1+6e+9e^2}{4} \right) + \frac{1}{2}mu^2 \left( \frac{9e^2-6e+1}{4} \right) \quad \dots (5m) \\ &= \frac{1}{8}mu^2(18e^2+2) = \frac{1}{4}mu^2(1+9e^2) \end{aligned}$$

$$\begin{aligned} \text{Loss in KE} &= \frac{10}{4}mu^2 - \frac{1}{4}mu^2(1+9e^2) \quad \dots (5m) \\ &= \frac{1}{4}mu^2(10-1-9e^2) \\ &= \frac{9}{4}mu^2(1-e^2) \end{aligned}$$

$$\begin{aligned} \text{Fraction lost} &= \frac{\frac{9}{4}mu^2(1-e^2)}{\frac{5}{2}mu^2} \\ \Rightarrow \frac{2}{5} \cdot \frac{9}{4}(1-e^2) &= \frac{27}{40} \\ \Rightarrow 4(1-e^2) &= 3 \\ \Rightarrow 1 &= 4e^2 \\ \Rightarrow e^2 = \frac{1}{4} \quad \Rightarrow e &= \frac{1}{2} \quad \dots (5m) \end{aligned}$$

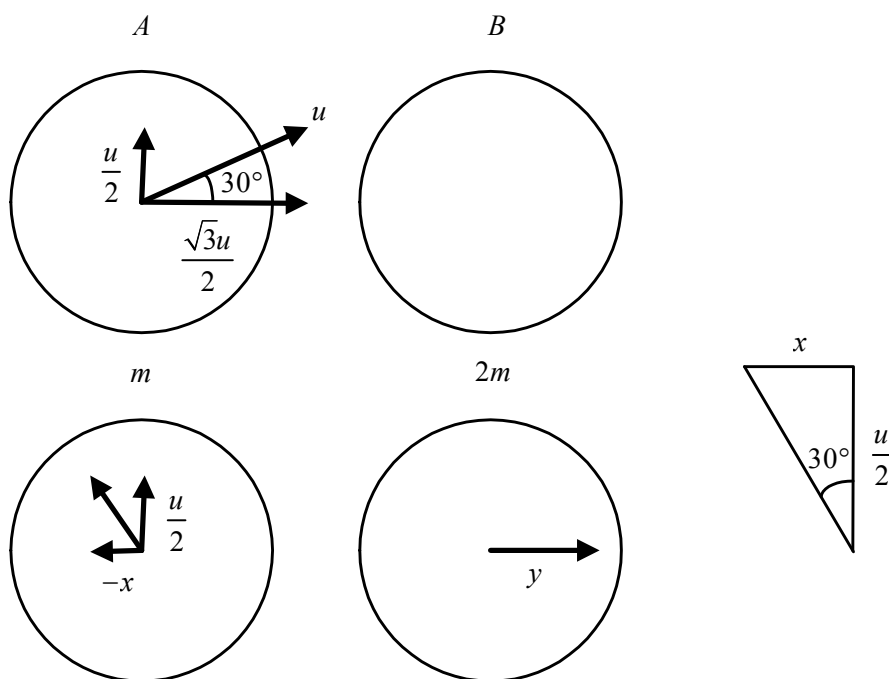


- (b) A smooth sphere A, of mass  $m$ , collides obliquely with a smooth sphere B, of mass  $2m$ , which is at rest.

Before the collision, A has a velocity of  $u$  in a direction which makes an angle of  $30^\circ$  with the line of centres.

If A is deflected through an angle of  $90^\circ$  by the collision, find the value of  $e$ , the coefficient of restitution.

(20)



$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{x}{\frac{u}{2}} \quad \Rightarrow \quad x = \frac{u}{2\sqrt{3}} \quad \dots (5m)$$

$$\text{PCM } \vec{i} : \quad m \left( \frac{-u}{2\sqrt{3}} \right) + 2my = m \left( \frac{\sqrt{3}u}{2} \right) \quad \dots (5m)$$

$$2y = \frac{\sqrt{3}u}{2} + \frac{u}{2\sqrt{3}}$$

$$2y = \frac{3u + u}{2\sqrt{3}} = \frac{2u}{\sqrt{3}}$$

$$y = \frac{u}{\sqrt{3}}$$

$$\text{NEL } \vec{i} : \quad \left( \frac{-u}{2\sqrt{3}} - \frac{u}{\sqrt{3}} \right) = -e \left( \frac{\sqrt{3}u}{2} \right) \quad \dots (5m)$$

$$-\frac{3u}{2\sqrt{3}} = -e \frac{\sqrt{3}u}{2}$$

$$\frac{\sqrt{3}u}{2} = e \frac{\sqrt{3}u}{2}$$

$$e = 1 \quad \dots (5m)$$

6. (a) A particle is performing simple harmonic motion of amplitude 0.8 m about a fixed point  $O$ .

$A$  and  $B$  are two points on the path of the particle such that  $|OA| = 0.6$  m and  $|OB| = 0.4$  m. The particle takes 2 seconds to travel from  $A$  to  $B$ .

Find, correct to two decimal places, the periodic time of the motion if

- (i)  $A$  and  $B$  are on the same side of  $O$  (15)

$$x = a \cos(\omega t + \epsilon)$$

$$x = 0.6, t = 0: \quad 0.6 = 0.8 \cos \epsilon$$

$$\cos \epsilon = 0.75$$

$$\epsilon = \cos^{-1} 0.75$$

$$\epsilon = 0.7227 \quad \dots (5\text{m})$$

$$x = 0.4, t = 2: \quad 0.4 = 0.8 \cos(2\omega + \epsilon)$$

$$\cos(2\omega + \epsilon) = 0.5$$

$$2\omega + \epsilon = \cos^{-1} 0.5$$

$$2\omega + 0.7227 = 1.0472$$

$$2\omega = 0.3245$$

$$\omega = 0.16225 \quad \dots (5\text{m})$$

Period:  $T = \frac{2\pi}{\omega}$

$$T = \frac{2\pi}{0.16225} = 38.73 \text{ s} \quad \dots (5\text{m})$$

- (ii)  $A$  and  $B$  are on opposite sides of  $O$ . (10)

$$x = a \cos(\omega t + \epsilon)$$

$$x = -0.4, t = 2: \quad -0.4 = 0.8 \cos(2\omega + \epsilon)$$

$$\cos(2\omega + \epsilon) = -0.5$$

$$2\omega + \epsilon = \cos^{-1}(-0.5)$$

$$2\omega + 0.7227 = 2.0944$$

$$2\omega = 1.3717$$

$$\omega = 0.68585 \quad \dots (5\text{m})$$

Period:  $T = \frac{2\pi}{\omega}$

$$T = \frac{2\pi}{0.68585} = 9.16 \text{ s} \quad \dots (5\text{m})$$

- (b) A particle P is moving on the inner surface of a smooth hemispherical bowl with centre  $O$  and radius  $2a$ .

The particle is describing a horizontal circle, centre  $C$ , with angular speed  $\sqrt{\frac{g}{a}}$ .

- Find (i) the magnitude of the force exerted on P by the surface of the bowl (20)

$$\omega = \sqrt{\frac{g}{a}}$$

The circular motion force is

$$\begin{aligned} m\omega^2 r &= m\left(\frac{g}{a}\right)r \\ &= \frac{mgr}{a} \end{aligned}$$

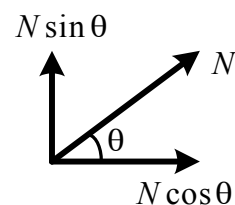
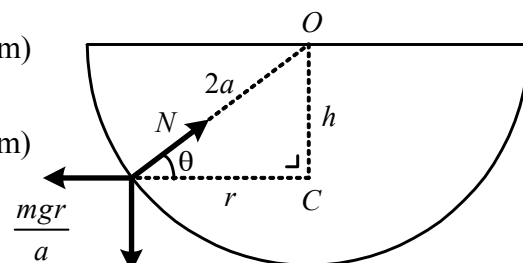
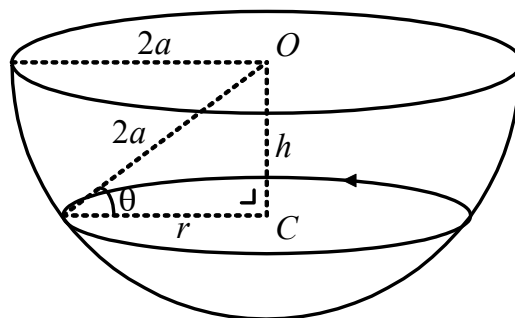
$$\uparrow = \downarrow : \quad N \sin \theta = mg \quad \dots 1 \quad \dots (5m)$$

$$\text{Circular : } N \cos \theta = \frac{mgr}{a} \quad \dots 2 \quad \dots (5m)$$

Also, from the triangle,

$$\cos \theta = \frac{r}{2a} \quad \dots 3 \quad \dots (5m)$$

$$\begin{aligned} 2, 3 : N\left(\frac{r}{2a}\right) &= \frac{mgr}{a} \\ N &= 2mg \end{aligned} \quad \dots (5m)$$



- (ii) the depth of  $C$  below  $O$ . (5)

$$1 : \quad 2mg \sin \theta = mg$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = 30^\circ$$

then

$$\frac{h}{2a} = \sin \theta$$

$$h = 2a\left(\frac{1}{2}\right)$$

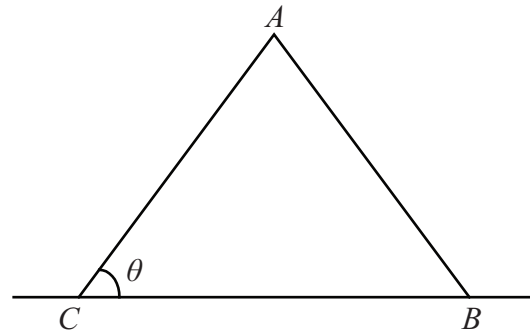
$$h = a \quad \dots (5m)$$

7. Two uniform rods,  $AB$  and  $AC$ , each of length  $2a$  and weight  $W$ , are smoothly jointed at  $A$ . The end  $C$  is freely hinged to a point on a rough horizontal plane.

The end  $B$  rests on the plane and is on the point of slipping.

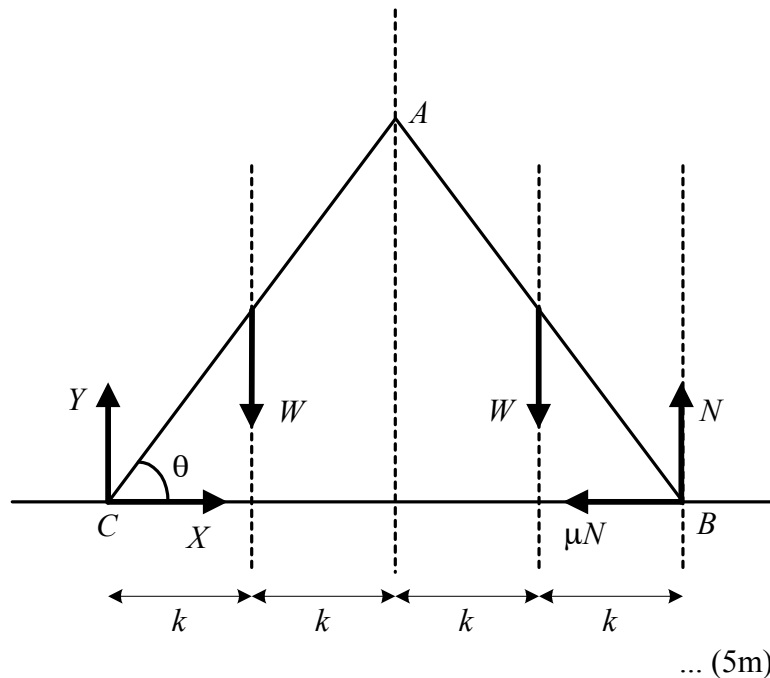
Both rods are in a vertical plane.

The coefficient of friction between  $B$  and the plane is  $\mu$  and the angle between  $AC$  and the horizontal is  $\theta$ .

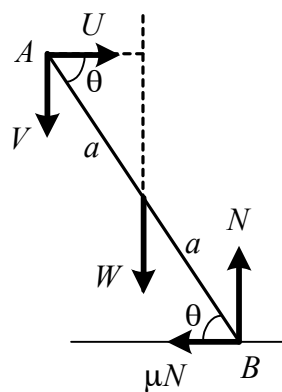


- (i) Show, on separate diagrams, all the forces acting on the structure  $ABC$  and on the rod  $AB$ . (10)

**Structure  $ABC$ :**



**Rod  $AB$ :**



... (5m)

(ii) Prove that  $\mu = \frac{1}{2 \tan \theta}$ . (15)

For structure  $ABC$ , moments about  $C$ :

$$W \cdot k + W \cdot 3k = N \cdot 4k \quad \dots (5m)$$

$$N = W \quad \dots 1$$

For rod  $AB$ , moments about  $A$ :

$$W \cdot a \cos \theta + \mu N \cdot 2a \sin \theta = N \cdot 2a \cos \theta \quad \dots (5m)$$

$$W + 2\mu N \tan \theta = 2N \quad \dots 2$$

1, 2:  $W + 2\mu W \tan \theta = 2W$

$$2\mu \tan \theta = 1$$

$$\mu = \frac{1}{2 \tan \theta} \quad \dots (5m)$$

(iii) Find, in terms of  $W$  and  $\mu$ , the magnitude and direction of the reaction force at the end  $B$  resting on the plane. (15)

The total reaction,  $R$ , at  $B$  is given by

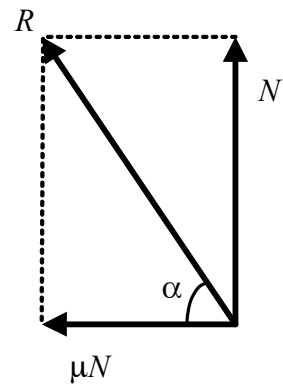
$$R = \sqrt{N^2 + (\mu N)^2} \quad \dots (5m)$$

$$= \sqrt{N^2 + \mu^2 N^2}$$

$$= \sqrt{N^2 (1 + \mu^2)}$$

$$= N \sqrt{1 + \mu^2}$$

$$= W \sqrt{1 + \mu^2} \quad \dots (5m)$$



The direction of this reaction force makes an angle  $\alpha$  with the horizontal, where

$$\tan \alpha = \frac{N}{\mu N} = \frac{1}{\mu}$$

i.e.  $\alpha = \tan^{-1} \frac{1}{\mu} \quad \dots (5m)$

(iv) Find, in terms of  $W$  and  $\theta$ , the magnitude of the reaction at the joint  $A$ . (10)

Rod  $AB$

$$\updownarrow: V + W = N$$

$$V + W = W$$

$$V = 0 \quad \dots (5m)$$

$$\leftrightarrow: U = \mu N$$

$$U = \mu W$$

or  $U = \frac{W}{2 \tan \theta} \quad \dots (5m)$

which is the magnitude of the reaction at the joint  $A$ .

8. (a) Prove that the moment of inertia of a uniform rod of mass  $m$  and length  $2l$  about an axis through its centre perpendicular to the rod is  $\frac{1}{3}ml^2$ . (20)

Standard Proof

Moment of mass element	... (5m)
Moment of body	... (5m)
Integral	... (5m)
Deduce	... (5m)

- (b) A uniform rod  $AB$  of mass  $m$  and length  $2l$  is free to rotate in a vertical plane about a horizontal axis through  $A$ . A particle of mass  $2m$  is attached to the rod at  $B$ .

The system is released from rest with  $B$  vertically above  $A$ .

- (i) Find the angular velocity when the system is next vertical. (20)

For the system,

$$I = I_{\text{rod}} + I_B$$

$$I_{\text{rod}} = \frac{4}{3}ml^2$$

$$I_B = mb^2$$

$$= (2m)(2l)^2$$

$$= 8ml^2$$

then

$$I = \frac{4}{3}ml^2 + 8ml^2$$

$$= \frac{28}{3}ml^2$$

... (5m)

**Position 1:**  $B$  above  $A$

$$KE_1 = 0$$

$$PE_1 = PE_{\text{rod}} + PE_B$$

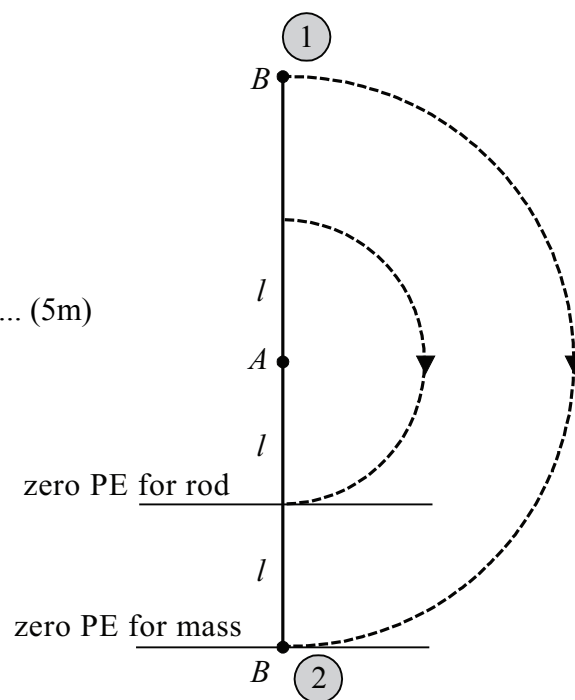
$$= mg(2l) + (2m)g(4l)$$

$$= 10mgl$$

**Position 2:**  $B$  below  $A$

$$KE_2 = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}\left(\frac{28}{3}ml^2\right)\omega^2$$



$$= \frac{14}{3} ml^2 \omega^2$$

$$PE_2 = 0$$

**PCE:**  $KE_2 + PE_2 = KE_1 + PE_1$

$$\frac{14}{3} ml^2 \omega^2 + 0 = 0 + 10mgl \quad \dots (5m, 5m)$$

$$7l\omega^2 = 15g$$

$$\omega^2 = \frac{15g}{7l}$$

$$\omega = \sqrt{\frac{15g}{7l}} \text{ rad s}^{-1} \quad \dots (5m)$$

At this point the mass at  $B$  falls off.

**(ii)** Find the height of  $B$  when it next comes to rest. **(10)**

For the rod after the mass falls off,

$$I = \frac{4}{3} ml^2$$

(Let the position of zero PE be when  $B$  is vertically below  $A$ .)

**Position 1:**  $B$  below  $A$ .

$$KE_1 = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} \left( \frac{4}{3} ml^2 \right) \left( \frac{15g}{7l} \right)$$

$$= \frac{10}{7} mgl$$

$$PE_1 = 0$$

**Position 2:** rod comes to rest.

Let the height of the centre of mass above the zero PE level be  $h_1$ .

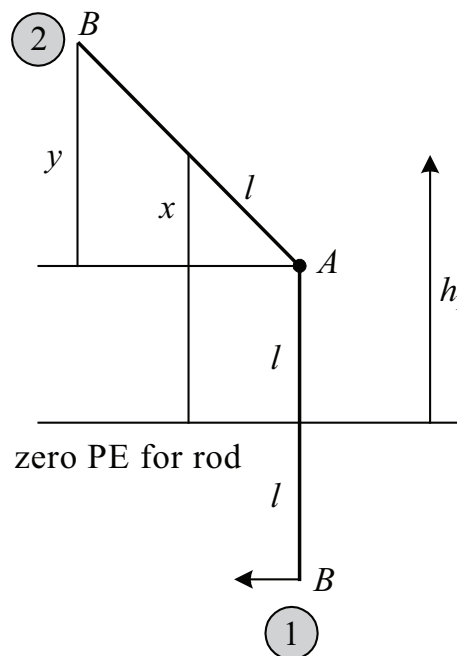
$$KE_2 = 0$$

$$PE_2 = mgh_1$$

**PCE:**  $KE_2 + PE_2 = KE_1 + PE_1$

$$0 + mgh_1 = \frac{10}{7} mgl + 0 \quad \dots (5m)$$

$$h_1 = \frac{10}{7} l$$



**Further answers overleaf**

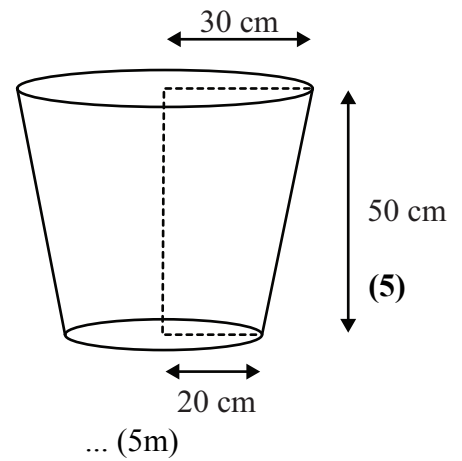
From the diagram,  $x = \frac{10}{7}l - l = \frac{3}{7}l$

then  $y = 2x = \frac{6}{7}l$

hence, at the highest point  $B$  is  $\frac{6}{7}l$  above the level of  $A$ . ... (5m)



9. (a) A bucket, in the shape of a frustum, has a height of 50 cm, a base of radius 20 cm and a top of radius 30 cm.



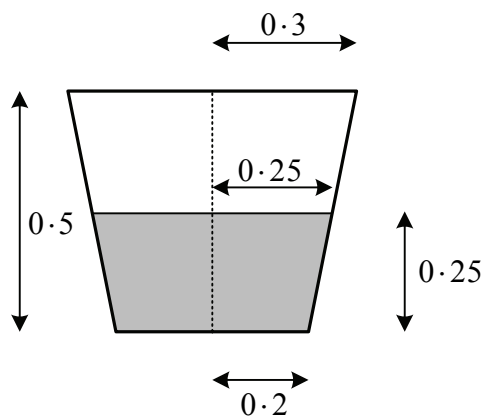
- Find, in terms of  $\pi$  and  $g$ ,  
 (i) the thrust on the base if the bucket is filled with water

Let  $T$  be the thrust on the base.

$$\begin{aligned} T &= \rho g h A \\ &= 1000(1)g(0.5)(\pi(0.2)^2) \\ &= 20\pi g \text{ N} \end{aligned}$$

- (ii) the thrust on the curved side if water is poured into the bucket to a depth of 25 cm.

(25)



Height of top of water is 25 cm = 0.25 m ... (5m)

Volume of water:

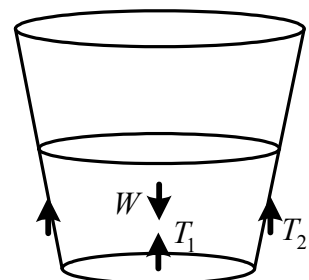
$$\begin{aligned} V &= \frac{1}{3}\pi(0.25)\left((0.25)^2 + (0.25)(0.2) + (0.2)^2\right) \\ &= \frac{61\pi}{4800} \text{ m}^3 \end{aligned}$$

... (5m)

Weight of water:

$$\begin{aligned} W &= \rho V g \\ &= 1000\left(\frac{61\pi}{4800}\right)g \\ &= 12.71\pi g \text{ N} \end{aligned}$$

... (5m)



**Further answers overleaf**

Thrust on base:

$$\begin{aligned} T_1 &= \rho g h A \\ &= 1000(1)g(0.25)(\pi(0.2)^2) \\ &= 10\pi g \text{ N} \end{aligned} \quad \dots (5\text{m})$$

Let  $T_2$  be the vertical thrust on the curved side. Then

$$\begin{aligned} T_1 + T_2 &= W \\ T_2 &= 12.71\pi g - 10\pi g \\ T_1 &= 2.71\pi g \text{ N} \end{aligned} \quad \dots (5\text{m})$$

- (b) A block of wood, of volume  $V$  and relative density  $s$ , floats in water.

A smaller block of metal, of volume  $V_1$  and relative density  $5s$ , is placed on top of the wood, such that the upper surface of the wooden block is in the free surface of the water.

Prove that  $\frac{V}{V_1} = \frac{5s}{1-s}$ .

Let  $B$  be the buoyancy,  $W$  the weight of the block of wood and  $W_1$  the weight of the block of metal. Let  $V$  be the volume of the wood and  $V_1$  be the volume of the metal.

$$\begin{aligned} B &= 1000(1)Vg \\ &= 1000gV \end{aligned}$$

and

$$\begin{aligned} \text{weight} &= W + W_1 \\ &= 1000(s)Vg + 1000(5s)V_1g \\ &= 1000sgV + 5000sgV_1 \end{aligned} \quad \dots (5\text{m})$$

In equilibrium:

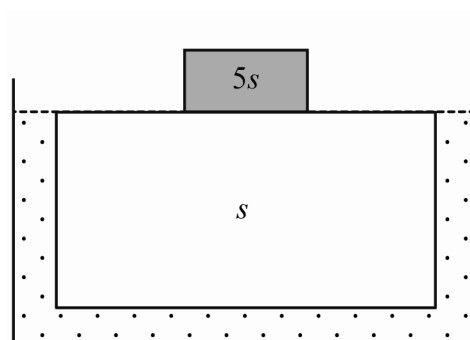
$$\begin{aligned} W + W_1 &= B \\ 1000sgV + 5000sgV_1 &= 1000gV \end{aligned} \quad \dots (5\text{m})$$

$$sV + 5sV_1 = V$$

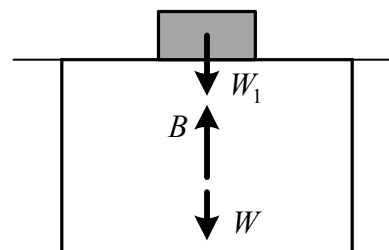
$$5sV_1 = V - sV$$

$$5sV_1 = (1-s)V$$

$$\frac{5s}{1-s} = \frac{V}{V_1} \quad \dots (5\text{m})$$



(20)



... (5m)

10. (a) Solve the differential equation

$$x \frac{dy}{dx} = y(1+x)$$

given that  $y = 3$  when  $x = 1$ .

(20)

$$x \frac{dy}{dx} = y(1+x)$$

$$\frac{dy}{y} = \frac{1+x}{x} dx$$

$$\int \frac{dy}{y} = \int \left( \frac{1}{x} + 1 \right) dx \quad \dots (5m)$$

$$\ln |y| = \ln |x| + x + c \quad \dots (5m)$$

$$\begin{aligned} \text{I.C.: } y = 3, x = 1 \\ \ln 3 = 0 + 1 + c \\ c = -1 + \ln 3 \quad \dots (5m) \end{aligned}$$

Unique solution:

$$\ln |y| = \ln |x| - x - 1 + \ln 3$$

$$\ln |y| = \ln |x| + \ln e^{-1-x} + \ln 3$$

$$\ln |y| = \ln 3 |x| e^{-1-x}$$

$$y = 3xe^{-1-x} = \frac{3x}{e^{1+x}} \quad \dots (5m)$$

- (b) A particle of mass  $m$  moves with a velocity  $v \text{ m s}^{-1}$  in a straight line through a medium in which the resistance to motion is  $mkv^3$ , where  $k$  is a constant. No other force acts on the body.

The velocity of the particle falls from  $15 \text{ m s}^{-1}$  to  $7.5 \text{ m s}^{-1}$  in a time of  $t_1$  seconds.

Show that the distance travelled in this time is  $10t_1 \text{ m}$ .

(30)

$$R = mkv^3$$

Newton's 2<sup>nd</sup> Law:

$$m \frac{d^2x}{dt^2} = -mkv^3$$

$$\frac{d^2x}{dt^2} = -kv^3$$

$$\frac{dv}{dt} = -kv^3 \quad \dots (5m)$$

**Further answers overleaf**

$$v^{-3} dv = -k dt$$

$$\int v^{-3} dv = -k \int dt$$

$$\frac{1}{-2} v^{-2} = -kt - c \quad \dots (5m)$$

$$\frac{1}{2v^2} = kt + c \quad \text{I.C.: } \underline{v=15, t=0}$$

$$\frac{1}{450} = c$$

Unique solution:

$$\frac{1}{2v^2} = kt + \frac{1}{450} \quad \dots (5m)$$

also,  $t = t_1$  when  $v = 7 \cdot 5$ :

$$\frac{1}{112 \cdot 5} = kt_1 + \frac{1}{450}$$

$$kt_1 = \frac{4}{450} - \frac{1}{450}$$

$$kt_1 = \frac{3}{450} = \frac{1}{150}$$

$$k = \frac{1}{150t_1} \quad \dots (5m)$$

then

$$v \frac{dv}{dx} = -kv^3$$

$$v^{-2} dv = -k dx$$

$$\int v^{-2} dv = -k \int dx \quad \dots (5m)$$

$$-\frac{1}{v} = -kx - d$$

$$\frac{1}{v} = kx + d \quad \text{I.C.: } \underline{v=15, x=0}$$

$$\frac{1}{15} = d$$

When  $v = 7 \cdot 5$ ,

$$\frac{1}{7 \cdot 5} = kx + \frac{1}{15}$$

$$kx = \frac{2}{15} - \frac{1}{15}$$

$$\frac{1}{150t_1} x = \frac{1}{15}$$

$$x = \frac{150t_1}{15}$$

$$x = 10t_1 \quad \dots (5m)$$

